KMA315 Analysis 3A: Solutions to Problems 3

1. Let:

(i) $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions;

(ii) $S = \{x \in \mathbb{R} : f(x) \geq g(x)\}$; and

(iii) $(x_n)_{n=0}^{\infty}$ be a sequence of points from S.

Show that if $\lim_{n\to\infty} x_n$ exists then $\lim_{n\to\infty} x_n \in S$. *(5 marks)*

Proof. Let:

(i) $h : \mathbb{R} \to \mathbb{R}$ be a continuous function;

(ii) $T = \{x \in \mathbb{R} : h(x) \geq 0\}$; and

(iii) $x \in \mathcal{C}(T)$ (ie. $h(x) < 0$).

Let $\delta = -\frac{h(x)}{2}$. It follows from h being continuous that there exists $\varepsilon > 0$ such that for each $x' \in (x - \varepsilon, x + \varepsilon)$, $h(x') \in (h(x) - \delta, h(x) + \delta) = (\frac{3h(x)}{2}, \frac{h(x)}{2})$, and hence $(x - \varepsilon, x + \varepsilon) \subseteq \mathcal{C}(T)$. Since there is an open ball around x entirely contained in $\mathcal{C}(T)$, x cannot be a limit point of T. Since x was any arbitrary point in $\mathcal{C}(T)$, all limit points of T must be in T, and hence T is closed.

Finally, letting $h = f - g$ (ie. $h(x) = (f - g)(x) = f(x) - g(x)$), we have $S = T$ (which is closed), and hence $\lim_{n\to\infty} x_n \in S$. □

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Kumudini's solution

Proof. Let:

(i) $h : \mathbb{R} \to \mathbb{R}$ be a continuous function;

(ii) $T = \{x \in \mathbb{R} : h(x) \geq 0\}$; and

(iii) $(x_n)_{n=0}^{\infty}$ be a convergent sequence of points from T.

Assume that $h(\lim_{n\to\infty} x_n) < 0$, ie. $\lim_{n\to\infty} x_n \notin T$. Since h is continuous, it follows from Proposition 4.3.13 in the typed notes that $\lim_{n\to\infty} h(x_n) = h(\lim_{n\to\infty} x_n)$.

Therefore, by the definition of the limit of a sequence, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|h(x_m) - h(\lim_{n\to\infty} x_n)| < \varepsilon$ for all $m \geq N$.

Let $\varepsilon = -\frac{h(\lim_{n\to\infty} x_n)}{2}$, then for each $m \geq N$ we have:

$$-\varepsilon = \frac{h(\lim_{n\to\infty} x_n)}{2} < h(x_m) - h(\lim_{n\to\infty} x_n) < -\frac{h(\lim_{n\to\infty} x_n)}{2} = \varepsilon$$
$$\Rightarrow \frac{3h(\lim_{n\to\infty} x_n)}{2} < h(x_m) < \frac{h(\lim_{n\to\infty} x_n)}{2} < 0.$$

But $h(x_m) < 0$ for all $m \geq N$ contradicts $(x_n)_{n=0}^{\infty}$ being a sequence of points from T. Therefore our assumption that $h(\lim_{n\to\infty} x_n) < 0$ cannot be true, and hence $\lim_{n\to\infty} x_n \in T$.

Finally, letting $h = f - g$ (ie. $h(x) = (f - g)(x) = f(x) - g(x)$), we have $S = T$, and hence $\lim_{n\to\infty} x_n \in S$. □

2. Let $f : [0,1] \to [0,1]$ be the function defined by

$$f(x) = \begin{cases} x & \text{when } x \in \mathbb{Q}\text{; and} \\ 1-x & \text{when } x \in \mathcal{C}(\mathbb{Q}). \end{cases}$$

Prove that:

(i) f assumes every value between 0 and 1 (ie. that f is surjective); *(1 mark)*

(ii) f is continuous only at $x = \frac{1}{2}$. *(2 marks)*

Proof. (i) For each $x \in [0,1]$, we trivially have $x = \begin{cases} f(x) & x \in \mathbb{Q}\text{; and} \\ f(1-x) & x \in \mathcal{C}(\mathbb{Q}). \end{cases}$

(ii) For each $\delta > 0$, $f(x) \in (f(\frac{1}{2}) - \delta, f(\frac{1}{2}) + \delta) = (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ is trivially satisfied for all $x \in (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$. Hence, f is continuous at $x = \frac{1}{2}$. Let $x \in \Big([0, \frac{1}{2}) \cup (\frac{1}{2}, 1]\Big) \cap \mathbb{Q}$. For each $\varepsilon > 0$, let $y \in (x - \varepsilon, x + \varepsilon) \cap \mathcal{C}(\mathbb{Q})$ (which necessarily exists since $\mathcal{C}(\mathbb{Q})$ are dense in $\mathbb{R}$), and let $\delta = \frac{|f(x) - f(y)|}{2}$. Then we have $y \in (x - \varepsilon, x + \varepsilon)$ such that $f(y) \notin (f(x) - \delta, f(x) + \delta)$. Since ε was arbitrary, f is discontinuous at x.

□

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ for all $x \in \mathbb{Q}$. Establish what value $f(x)$ takes for irrational values of x. *(3 marks)*

Proof. Let $x \in \mathcal{C}(\mathbb{Q})$. Suppose $f(x) \neq 0$ and let $\delta = \frac{|f(x)|}{2}$. Since $\mathbb{Q}$ are dense in $\mathbb{R}$, for each ε there exists $x' \in (x - \varepsilon, x + \varepsilon) \cap \mathbb{Q}$, which satisfies $f(x') = 0 \notin (f(x) - \delta, f(x) + \delta)$. Hence under the assumption that $f(x) \neq 0$ we would have that f is discontinuous at x, which contradicts f being continuous. Therefore we must also have $f(x) = 0$ for all irrational x. $\square$

4. Let $(f_n)_{n=0}^{\infty}$ be the sequence of real-valued functions on $\mathbb{R}$ where for each $n \in \mathbb{N}$,

$$f_n(x) = x + \frac{1}{n} \text{ for all } x \in \mathbb{R}.$$

Establish that:

(i) $(f_n)_{n=0}^{\infty}$ converges uniformly on $\mathbb{R}$; *(2 marks)*

(ii) $(f_n^2)_{n=0}^{\infty}$ <u>does not</u> converge uniformly on $\mathbb{R}$. *(3 marks)*
Note: for each $n \in \mathbb{N}$, $f_n^2(x) = [f_n(x)]^2$ for all $x \in \mathbb{R}$.

Proof. (i) For each $\varepsilon > 0$, there trivially exists $N \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ for all $n \geq N$. In which case we have $|f_n(x) - f(x)| = \frac{1}{n} < \varepsilon$ for all $n \geq N$ and $x \in \mathbb{R}$. Hence $(f_n)_{n=0}^{\infty}$ converges uniformly to f.

(ii) For each $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $f_n^2(x) = (x+\frac{1}{n})^2 = x^2+\frac{2}{n}x+\frac{1}{n^2}$ which is a 'happy-face' quadratic with a single root at $-\frac{1}{n}$. It is trivially the case that for each $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n^2(x) = x^2$, and hence that f_n^2 converges pointwise to f^2. However for each $\varepsilon > 0$ and $n \in \mathbb{N}$:

$$\begin{aligned} f_n(x) - f(x) &> \varepsilon \\ \Rightarrow x^2 + \frac{2}{n}x + \frac{1}{n^2} - x^2 &> \varepsilon \\ \Rightarrow \frac{2}{n}x + \frac{1}{n^2} &> \varepsilon \\ \Rightarrow \frac{2}{n}x &> \varepsilon - \frac{1}{n^2} \\ \Rightarrow x &> \frac{n^2\varepsilon - 1}{2n} \end{aligned}$$

Hence for each $x > \frac{n^2\varepsilon-1}{2n}$, $|f_n(x) - f(x)| > \varepsilon$, and hence f_n^2 does not converge uniformly.

□

5. Let $(f_n)_{n=0}^{\infty}$ be the sequence of real-valued functions on $[0,1]$ where for each $n \in \mathbb{N}$,

$$f_n(x) = x^n \text{ for all } x \in [0,1].$$

(i) Establish whether $(f_n)_{n=0}^{\infty}$ converges pointwise; *(1 mark)*

(ii) if it does, find the pointwise limit of $(f_n)_{n=0}^{\infty}$. *(1 mark)*

For each $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & 0 \le x < 1; \\ 1 & x = 1. \end{cases}$

Hence the pointwise limit of $(f_n)_{n=0}^{\infty}$ is $f : [0,1] \to [0,1]$ where $f(x) = \begin{cases} 0 & 0 \le x < 1; \\ 1 & x = 1. \end{cases}$